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Stabilized Continuation Method for Solving Optimal Control Problems

Toshiyuki Ohtsuka* and Hironori Fujii†
Tokyo Metropolitan Institute of Technology, Tokyo 191, Japan

A continuation method is discussed in this paper for solving a large class of optimal control problems with general boundary conditions and nondifferential constraints. The method converts the optimal control problems into initial-value problems of finite-dimensional ordinary differential equations that can be solved with existing algorithms for numerical integration. The present continuation method is stabilized in the sense that stabilization techniques are introduced to avoid accumulation of error in the integration process of the differential equations. It is shown that the stabilization of the continuation method is equivalent to control of linear systems, and several stabilizing techniques are considered. Furthermore, the multiplier method (augmented Lagrangian method) of continuous type is developed for the continuation method to solve optimal control problems with boundary constraints. The effectiveness of the present method is demonstrated for an example with initial and terminal constraints.

I. Introduction

OPTIMAL control problems have a large class of applications in aerospace engineering. Optimal control problems result in nonlinear two-point boundary-value problems (TPBVPs), and the difficulties in solving the nonlinear TPBVP lead to cumbersome numerical computation. Although many computational methods have been proposed, modification of the existing methods and development of new methods should yet be exploited earnestly to obtain exact solutions successfully.

A continuation method is developed in this paper for solving the optimal control problems. The principle of continuation methods is that a given problem is embedded into a family of problems parameterized by a parameter, and the solution is traced varying the parameter from the initial problem with a known solution to the original problem. Fundamental theory of the continuation method is summarized in Refs. 1 and 2. Roberts and Shipman[3,4] apply the continuation method to the shooting method for solving TPBVPs. In their method, a sequence of TPBVPs are solved successively for a sequence of time spans increasing stepwise. The solution for a certain time span is taken as the initial guess of the solution for the next time span in order to overcome the numerical difficulty in the shooting method, which is quite sensitive to the initial guess of the solution. The stepwise continuation in time span is studied further and modified by Orava and Lautala.[5] In Ref. 6, the boundary condition in TPBVPs is embedded into a family of boundary conditions in order to apply the continuation method. Parameters are varied sequentially with small steps and the corresponding problem is solved by using some iterative method in Refs. 3–6. In contrast to Refs. 3–6, the parameter is increased continuously, resulting in an initial-value problem of a differential equation, and no iterative method is involved in Ref. 7.

The continuation method is also studied and modified from the standpoint of the differential geometry. Chow et al.[8] present a solution method of nonlinear equations that is constructive with probability 1. Although their method is similar to the classical continuation method, the parameterized solution is traced in their method with respect to the arc length of the solution curve, not the parameter that parameterizes the problem. Their method is often called the homotopy method in order to distinguish it from the classical continuation method. The homotopy method is applied successfully to a class of nonlinear TPBVPs,[9,10] constrained nonlinear programming problems,[11,12] and so on.

In spite of some work on the application of the continuation method to TPBVPs, the general formulation has not been presented for the continuation method applied to a broad class of optimal control problems specifically. The first objective of this paper is to formulate and analyze the continuation method for optimal control problems of a general problem setup. Our formulation of the continuation method is similar to the approach in Ref. 7 in the sense that optimal control problems are converted to initial-value problems of finite-dimensional ordinary differential equations. One does not have to solve TPBVPs directly, and existing algorithms for numerical integration are available in solving the initial-value problems. Although we focus our attention on optimal control problems rather than general TPBVPs, we treat the general form of optimal control problems, including general boundary conditions and nondifferential constraints, which are not discussed in the previous work. The parameter for the continuation method can be chosen with freedom, because of the generality of the formulation.

The second objective of this paper is to propose stabilization techniques for the continuation method. That is to say, stabilizing techniques are introduced to avoid accumulation of error in solving the initial-value problems associated with the continuation method. Whereas the conventional method utilizes such an iterative algorithm as the Newton method in order to correct the error,[2,13] this paper modifies the continuation method so that the error attenuates as the integration proceeds. Regarding the continuation method as stabilization of a linear dynamical system, several stabilizing techniques are proposed.

Furthermore, selection of the parameter to be varied and treatment of the boundary constraints are discussed in this paper. The multiplier method (augmented Lagrangian method) and one of the proposed stabilizing techniques are considered for the continuation method in problems with boundary constraints. The multiplier is updated by a differential equation in the present multiplier method, whereas it is updated stepwise in the conventional multiplier method. Stability characteristics of the present multiplier method is analyzed by Lyapunov's method. A numerical example shows the aspects of the stabilized continuation method.

II. Formulation of Stabilized Continuation Method

A. General Formulation of Continuation Method for Optimal Control Problems

We consider an optimal control problem parameterized by a parameter $s \in [0, s_f]$. The objective is to obtain the optimal control for $s = s_f$. Even if there are several parameters s_i to be varied, they are parameterized by the one parameter as $s_i(s)$. We call the

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*Graduate Student, Department of Aerospace Engineering, 6-6 Asahigaoka, Hino. Student Member AIAA.

†Professor, Department of Aerospace Engineering, 6-6 Asahigaoka, Hino. Member AIAA.

parameter s the continuation parameter in order to distinguish it from free parameters to be optimized. The parameterized problem is to obtain the optimal control input that minimizes the performance index:

$$J_s = \eta[x(0), p; s] + \varphi[x(1), p; s] + \int_0^1 L[x(t), u(t), t, p; s]\,\mathrm{d}t \tag{1}$$

subject to the constraints

$$\dot{x}(t) = f[x(t), u(t), t, p; s], \qquad f \in \boldsymbol{R}^n \tag{2}$$

$$\chi[x(0), p; s] = 0, \qquad \chi \in \boldsymbol{R}^{n_0};\ n_0 \le n \tag{3}$$

$$\psi[x(1), p; s] = 0, \qquad \psi \in \boldsymbol{R}^{n_1};\ n_1 \le n \tag{4}$$

$$C[x(t), u(t), t, p; s] = 0, \qquad C \in \boldsymbol{R}^{n_c};\ n_c \le m \tag{5}$$

where $x \in \boldsymbol{R}^n$ denotes the state of the dynamical system, $u \in \boldsymbol{R}^m$ the control input, and $p \in \boldsymbol{R}^{n_p}$ free parameters to be optimized. The arguments of functions are often omitted hereafter. The functions are assumed to be differentiable as many times as necessary. The terminal time of the problem is normalized to 1 without loss of generality. The terminal time is regarded as a free parameter if the terminal time is not specified. Inequality constraints on the state and control can be converted to the equality constraints in Eq. (5) by introducing dummy variables. The augmented performance index is defined in order to apply the calculus of variation:

$$\bar{J}_s = X + Y + \int_0^1 [L + \lambda^T(f - \dot{x}) + \rho^T C]\,\mathrm{d}t \tag{6}$$

where

$$X = \eta + \mu^T \chi \tag{7}$$

$$Y = \varphi + \nu^T \psi \tag{8}$$

and $\lambda(t)$, $\rho(t)$, ν, and μ are Lagrange multipliers of appropriate dimensions. The Hamiltonian H is defined for the optimal control problem as

$$H = L + \lambda^T f + \rho^T C \tag{9}$$

and for a fixed s, the necessary conditions of the optimality are obtained from the calculus of variation as follows:

$$\dot{\lambda} = -H_x^T \tag{10}$$

$$H_u = 0 \tag{11}$$

$$X_x^T + \lambda(0) = 0 \tag{12}$$

$$Y_x^T - \lambda(1) = 0 \tag{13}$$

$$X_p + Y_p + \int_0^1 H_p\,\mathrm{d}t = 0 \tag{14}$$

The above equations have to be satisfied subject to the constraints in Eqs. (2–5) and constitute a TPBVP. The control input $u(t)$ and the Lagrange multiplier $\rho(t)$ are determined from Eqs. (5) and (11) for a pair of $x(t)$ and $\lambda(t)$. The state $x(t)$ and the costate $\lambda(t)$ are functions of the initial conditions and the free parameter p, since they are obtained integrating the Euler-Lagrange equation [Eqs. (2) and (10)], starting from the initial conditions $x(0)$ and $\lambda(0)$. With the conditions in Eqs. (2), (5), (10), and (11) satisfied, the TPBVP is equivalent to a nonlinear equation:

$$F[z(s), s] = 0, \quad z, F \in \boldsymbol{R}^{(2n+np+n_0+n_1)} \tag{15}$$

where the function F and the vector z are defined as

$$F = \begin{bmatrix} X_x^T + \lambda(0) \\ Y_x^T - \lambda(1) \\ X_p + Y_p + \int_0^1 H_p\,\mathrm{d}t \\ \chi[x(0), p; s] \\ \psi[x(1), p; s] \end{bmatrix}, \qquad z = \begin{bmatrix} x(0) \\ \lambda(0) \\ p \\ \mu \\ \nu \end{bmatrix} \tag{16}$$

Note that $x(1)$ and $\lambda(1)$ are determined implicitly from z by the Euler-Lagrange equations. The function F and the solution vector z depend on the continuation parameter s, because the optimal control problem depends on s. The differential equation

$$\frac{\mathrm{d}F[z(s), s]}{\mathrm{d}s} = 0 \tag{17}$$

holds along the curve of solution $(z(s), s)$ for Eq. (15), and the differential equation (Davidenko equation) for $z(s)$ is obtained from Eq. (17) as

$$\frac{\mathrm{d}z}{\mathrm{d}s} = -F_z^{-1} F_s \tag{18}$$

where the nonsingularity of the Jacobian matrix F_z is assumed. If the solution of Eq. (15) is known for $s = 0$ and if the solution is required at $s = s_f$, then $z(s_f)$ is calculated by integrating Eq. (18) starting from the known initial condition $z(0)$ to $s = s_f$. The optimal control problem is solved as an initial-value problem for a finite-dimensional ordinary differential equation, for which established numerical algorithms exist.

The major difficulty occurs in the above continuation method when the Jacobian matrix F_z becomes singular in integrating Eq. (18). When the Jacobian matrix F_z is singular, the continuation parameter is changed from s to the arc length σ of the curve of the solution $(z(\sigma), s(\sigma))$ in order to remove the singularity, and the differential equation (18) is replaced by[2]

$$[F_z \quad F_s] \begin{bmatrix} \dfrac{\mathrm{d}z}{\mathrm{d}\sigma} \\ \dfrac{\mathrm{d}s}{\mathrm{d}\sigma} \end{bmatrix} = 0, \qquad \left\| \begin{bmatrix} \dfrac{\mathrm{d}z}{\mathrm{d}\sigma} \\ \dfrac{\mathrm{d}s}{\mathrm{d}\sigma} \end{bmatrix} \right\| = 1 \tag{19}$$

When the matrix $[F_z\ F_s]$ has full rank even if F_z is singular, the derivatives of z and s with respect to the arc length σ are determined uniquely from Eq. (19) and a continuity condition by rather complicated procedures.[13] If the matrix $[F_z\ F_s]$ does not have full rank on the set of solution for $F(z, s) = 0$, one can avoid this difficulty by replacing the function F in Eq. (19) with the following function F':

$$F'(z, s; a) = sF(z, s) - (1 - s)(z - a) = 0, \qquad s \in [0, 1] \tag{20}$$

where a denotes a supplementary parameter. The parameterized Sard theorem[8] implies that, for almost all a with respect to Lebesgue measure, there is a zero curve of F' for $s \in [0, 1)$ along which $[F_z'\ F_s']$ has full rank. A similar argument as in Refs. 9 and 13 shows that if the zero curve of F' for $s \in [0, 1)$ is bounded, it has an accumulation point $(\bar{z}, 1)$, where $F(\bar{z}, 1) = 0$. Furthermore if $F_z(\bar{z}, 1)$ is nonsingular, then the zero curve has finite arc length.[9,13]

One of advantages of the continuation method is that a large class of problems is solved by tracing the solution curve of $F(z, s) = 0$ without recourse to any iterative approximation methods that require a good initial guess of the solution. One drawback of the continuation method is that one has to compute F_z and F_s many times in order to trace the solution curve.

B. Stabilization Techniques in Continuation Method

Since error in the solution may accumulate through the numerical integration process, some techniques[2,13,14] are developed to correct the error in the literature. Although such an iterative algorithm as the

Newton method is often used to correct the error,[2,13] it is more efficient from the computational point of view to modify the differential equation so that the error attenuates as the integration proceeds. Kabamba et al.[14] introduce such a restoration term in Eq. (17) as

$$\frac{dF}{ds} = -\alpha F \tag{21}$$

where α is a positive scalar constant. All elements of $F[z(s), s]$ converges zero exponentially, and therefore $z(s)$ converges to the correct solution. Such a stabilizing term as in Eq. (21) is also discussed by Baumgarte[15] for the simulation of mechanical systems and by Branin[16] for solving nonlinear equations, although they do not intend the continuation method itself and the continuation parameter is not included explicitly in their problems.

One of objectives in this paper is further extension of the stabilizing term. In general, the stabilization of the solution to the correct solution can be viewed as a class of control problems of a linear system expressed as

$$\frac{dF}{ds} = \nu \tag{22}$$

where ν denotes the input to the system for stabilization. Deviation of F from the origin, $F = 0$, is corrected through the integration process by a stabilizing control input. The vector $z(s)$ is obtained by integrating the differential equation

$$\frac{dz}{ds} = F_z^{-1}(\nu - F_s) \tag{23}$$

When the Jacobian matrix F_z is singular, the continuation parameter is changed from s to the arc length σ of the curve $(z(\sigma), s(\sigma))$ in order to remove the singularity, and the differential equation (23) is replaced by

$$\begin{bmatrix} \dfrac{dz}{d\sigma} \\ \dfrac{ds}{d\sigma} \end{bmatrix} = [F_z \quad F_s]^{+}\nu + r \tag{24}$$

where $[F_z\ F_s]^{+}$ denotes the Moore-Penrose inverse of $[F_z\ F_s]$, and a vector r is determined by

$$[F_z \quad F_s]r = 0, \qquad \|r\| = 1 \tag{25}$$

The vector r is determined uniquely from Eq. (25) and a continuity condition with respect to the arc length σ. The vector r is along the curve $F =$ const, and the vector $[F_z\ F_s]^{+}\nu$ is orthogonal with r. The same arguments hold on the integration of Eq. (24) as the previous section.

State Feedback

Many types of stabilizing input are possible, and several types are discussed in this paper. The most simple stabilizing input is a state feedback, expressed as

$$\nu(s) = AF[z(s), s] \tag{26}$$

where A is a stable matrix. Equation (26) is a direct generalization of Eq. (21). One can also choose the input $\nu(s)$ as a linear-quadratic regulator that minimizes the following measure of error:

$$J_e = \frac{1}{2}F^T(s_f)S_f F^T(s_f) + \frac{1}{2}\int_0^{s_f}\left(F^T QF + \nu^T R\nu\right) ds \tag{27}$$

where S_f and $Q(s)$ are positive semidefinite matrices and $R(s)$ a positive definite matrix. The input $\nu(s)$ is then given as a state feedback:

$$\nu(s) = -R^{-1}(s)S(s)F[z(s), s] \tag{28}$$

where $S(s)$ is the solution of the corresponding Riccati equation:

$$\dot{S}(s) = S(s)R^{-1}(s)S(s) - Q(s), \qquad S(s) = S_f \tag{29}$$

When R and Q are constant, the solution $S(s)$ of Eq. (29) is obtained explicitly by substituting $S = -\dot{\Sigma}\Sigma^{-1}R$ and by solving a differential equation for Σ. The resultant solution $S(s)$ is expressed as

$$\begin{aligned} S(s) = -U\Big[&e^{-U(s_f-s)}\left(U + S_f R^{-1}\right)^{-1} \\ &- e^{U(s_f-s)}\left(U - S_f R^{-1}\right)^{-1}\Big]\Big[e^{-U(s_f-s)}\left(U + S_f R^{-1}\right)^{-1} \\ &+ e^{-U(s_f-s)}\left(U - S_f R^{-1}\right)^{-1}\Big]^{-1} R \end{aligned} \tag{30}$$

$$U = (QR^{-1})^{1/2} \tag{31}$$

In a steady-state case ($s_f \to \infty$) Eq. (28) reduces to a state feedback with a constant gain matrix.

Open-Loop Control

When the stabilizing term is given in the form of such a state feedback as in Eqs. (26) or (28), the closed-loop system reduces to a linear system again. One can introduce an additive open-loop control to the closed-loop system in order to set F to zero at a prescribed value of s. The system is expressed as

$$\frac{dF}{ds} = AF + \nu_a \tag{32}$$

where A denotes the gain matrix of the state feedback and ν_a the additive control input. Even if the function F is not zero at $s = 0$, it is set to zero at $s = s_f$ by the open-loop control input:

$$\nu_a(s) = -\Phi^T(0, s)W^{-1}(0, s_f)F[z(0), 0] \tag{33}$$

where $\Phi(0, s)$ denotes the transition matrix of the closed-loop system by the state feedback and $W(0, s)$ the controllability Gramian matrix defined as[17]

$$W(s_1, s_2) = \int_{s_1}^{s_2} \Phi(s_1, s)\Phi^T(s_1, s)\, ds \tag{34}$$

The controllability Gramian matrix $W(s_1, s_2)$ is nonsingular for any pair of (s_1, s_2) $(s_2 > s_1)$ because the transition matrix $\Phi(s_1, s)$ is nonsingular. Therefore the system of Eq. (32) is controllable by the additive input ν_a, and it is checked easily that F is set to zero by ν_a given as an open-loop control in Eq. (33). The open-loop control is useful when the solution cannot be obtained easily for $F(z, 0) = 0$.

Tracking Control Law

Another type of stabilizing input can be designed as a tracking control law. Given a reference input $\nu_{\text{ref}}(s)$ and the corresponding reference trajectory $F_{\text{ref}}(s)$ $[F_{\text{ref}}(s_f) = 0]$, the stabilizing input can be a tracking control law:

$$\nu(s) = \nu_{\text{ref}}(s) + K_t\{F[z(s), s] - F_{\text{ref}}(s)\} \tag{35}$$

where K_t is a gain matrix determined appropriately. The error between actual and reference trajectories $(F - F_{\text{ref}})$ is governed by the differential equation

$$\frac{d}{ds}(F - F_{\text{ref}}) = K_t(F - F_{\text{ref}}) \tag{36}$$

The error $F - F_{\text{ref}}$ attenuates as the integration proceeds if K_t is stable. It is shown from Eq. (36) that the matrix K_t is regarded as a gain matrix of state feedback. Therefore K_t may be determined as an optimal feedback gain of a linear quadratic regulator.

Nonlinear Feedback

The stabilizing term can also be designed as a nonlinear feedback. Introducing a candidate of a Lyapunov function $V(F)$, a stabilizing input is given by

$$\nu = -Q\left(\frac{dV}{dF}\right)^T \tag{37}$$

where Q is a positive definite matrix. The function $V(F)$ is monotonously decreasing with respect to s since

$$\frac{\mathrm{d}V}{\mathrm{d}s} = -\frac{\mathrm{d}V}{\mathrm{d}F} Q \left(\frac{\mathrm{d}V}{\mathrm{d}F}\right)^T \le 0 \tag{38}$$

The Lyapunov theorem[18] implies that the origin, $F = 0$, is asymptotically stable in the sense of Lyapunov, if the derivative $\mathrm{d}V/\mathrm{d}F$ is not the null vector except for the origin.

C. Variation of Optimal Solution with Respect to Continuation Parameter

In order to integrate Eq. (23) or (24), it is necessary to evaluate the derivatives of the function F with respect to z and s. They are evaluated from the calculus of variations, since the derivative $\mathrm{d}z/\mathrm{d}s$ is expressed as variations of the optimal solution with respect to s, i.e.,

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \begin{bmatrix} \frac{\delta x^T(0)}{\mathrm{d}s} & \frac{\delta\lambda^T(0)}{\mathrm{d}s} & \frac{\mathrm{d}p^T}{\mathrm{d}s} & \frac{\mathrm{d}\mu^T}{\mathrm{d}s} & \frac{\mathrm{d}\nu^T}{\mathrm{d}s} \end{bmatrix}^T \tag{39}$$

where $\delta x(t)$ denotes the variation of the optimal trajectory caused by the infinitesimal variation in s, $\mathrm{d}s$, and so forth.

Variations of $u(t)$ and $\rho(t)$ are expressed in terms of the state, costate, and other parameters as follows:

$$\begin{bmatrix} \delta u \\ \delta\rho \end{bmatrix} = J \begin{bmatrix} \delta x \\ \delta\lambda \end{bmatrix} + K\,\mathrm{d}p + M\,\mathrm{d}s \tag{40}$$

where

$$[J \quad K \quad M] = -\begin{bmatrix} H_{uu} & C_u^T \\ C_u & 0 \end{bmatrix}^{-1} \begin{bmatrix} H_{ux} & f_u^T & \vdots & H_{up} & \vdots & H_{us} \\ C_x & 0 & \vdots & C_p & \vdots & C_s \end{bmatrix} \tag{41}$$

The matrix in the right-hand side of Eq. (41) is partitioned compatibly with sizes of J, K, and M. We assume

$$\det \begin{bmatrix} H_{uu} & C_u^T \\ C_u & 0 \end{bmatrix} \neq 0 \tag{42}$$

which is required commonly in such an existing second-order algorithm for optimal control problems as the modified quasilinearization algorithm (MQA).[19,20] The variations in the state and costate are governed by the following variational equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \delta x \\ \delta\lambda \end{bmatrix} = A_c \begin{bmatrix} \delta x \\ \delta\lambda \end{bmatrix} + B_c\,\mathrm{d}p + C_c\,\mathrm{d}s \tag{43}$$

where

$$[A_c \quad B_c \quad C_c] = \begin{bmatrix} f_x & 0 & \vdots & f_p & \vdots & f_s \\ -H_{xx} & -f_x^T & \vdots & -H_{xp} & \vdots & -H_{xs} \end{bmatrix} + \begin{bmatrix} f_u & 0 \\ -H_{xu} & -C_u^T \end{bmatrix} [J \quad K \quad M] \tag{44}$$

The variations of the state and costate depend linearly on the variations at initial conditions and parameters as

$$\begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} \delta x(0) \\ \delta\lambda(0) \end{bmatrix} + \Psi(t)\,\mathrm{d}p + \Omega(t)\,\mathrm{d}s \tag{45}$$

where the matrices $\Phi(t)$, $\Psi(t)$, and $\Omega(t)$ satisfy the following linear differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\Phi \quad \Psi \quad \Omega] = A_c[\Phi \quad \Psi \quad \Omega] + [0 \quad B_c \quad C_c] \tag{46}$$

with the initial condition

$$[\Phi(0) \quad \Psi(0) \quad \Omega(0)] = [I \quad 0 \quad 0] \tag{47}$$

The variations of the terminal state and costate are expressed by Eq. (45) in terms of variations of the initial state, the initial costate, and the parameters. The variations in the boundary conditions in Eqs. (3), (4), and (12–14) and the use of Eq. (45) yield the derivatives of the function $F(z, s)$ as follows:

$$F_z = \begin{bmatrix} X_{xx} & I & X_{xp} & \chi_x^T & 0 \\ S_1 & S_2 & S_3 & 0 & \psi_x^T \\ T_1 & T_2 & T_3 & \chi_p^T & \psi_p^T \\ \chi_x & 0 & \chi_p & 0 & 0 \\ U_1 & U_2 & U_3 & 0 & 0 \end{bmatrix}, \qquad F_s = \begin{bmatrix} X_{xs} \\ S_4 \\ T_4 \\ \chi_s \\ U_4 \end{bmatrix} \tag{48}$$

where

$$[[S_1 \quad S_2] \quad S_3 \quad S_4] = [Y_{xx} \quad -I][\Phi(1) \quad \Psi(1) \quad \Omega(1)] + [0 \quad Y_{xp} \quad Y_{xs}] \tag{49}$$

$$[[T_1 \quad T_2] \quad T_3 \quad T_4] = [Y_{px} \quad 0][\Phi(1) \quad \Psi(1) \quad \Omega(1)] + [[X_{px} \quad 0] \quad X_{pp} + Y_{pp} \quad X_{ps} + Y_{ps}] + \int_0^1 [[R_1 \quad R_2] \quad R_3 + H_{pp} \quad R_4 + H_{ps}]\,\mathrm{d}t \tag{50}$$

$$[[R_1 \quad R_2] \quad R_3 \quad R_4](t) = \left(\left[H_{px} \quad f_p^T\right] + \left[H_{pu} \quad C_p^T\right]J\right) \times [\Phi \quad \Psi \quad \Omega] + \left[H_{pu} \quad C_p^T\right][0 \quad K \quad M] \tag{51}$$

$$[[U_1 \quad U_2] \quad U_3 \quad U_4] = \psi_x[I \quad 0] \times [\Phi(1) \quad \Psi(1) \quad \Omega(1)] + [0 \quad \psi_p \quad \psi_s] \tag{52}$$

The matrices are partitioned on both sides of Eqs. (48–52) compatibly with each other. The number of columns of T_i, R_i, and U_i ($i = 1, 2$) is half of that of $\Phi(1)$. The vector $z(s)$ is obtained by integrating Eqs. (23) or (24) where the matrices F_z and F_s are given by Eq. (48). The desired optimal trajectory is obtained from $z(s_f)$ by integrating the Euler-Lagrange equation [Eqs. (2) and (10)] with the optimal control input determined by Eqs. (5) and (11). Computation of F, F_z, and F_s requires integration of $H_p(t)$, $\mathrm{d}\Phi(t)/\mathrm{d}t$, $R_1(t)$, $H_{pp}(t)$, and other quantities that depend on $x(t)$, $\lambda(t)$, $u(t)$, and $\rho(t)$. Since integration of those quantities can be carried out in parallel with integration of the Euler-Lagrange equation by using the Runge-Kutta or other methods, functions $x(t)$, $\lambda(t)$, $u(t)$, and $\rho(t)$ do not have to be stored over the whole interval [0, 1]. Therefore the continuation method requires less data storage than such existing algorithms as the sequential conjugate gradient-restoration algorithm (SCGRA)[21] and the MQA, which need to store those functions.

III. Discussion on Problems with General Boundary Conditions

A. Problems with Given Initial State and Terminal Constraint

The feedback-type stabilizing techniques do not bring a nonzero F to the origin exactly at a prescribed value of s, since they guarantee only asymptotic stability. Therefore $z(0)$ should be given so as to satisfy $F[z(0), 0] = 0$, unless F is replaced with F' defined in Eq. (20) or such an open-loop control is introduced as Eq. (33). However, it is often difficult to find a solution that satisfies both initial and terminal conditions in an optimal control problem, even if the problem is simplified at $s = 0$. Furthermore, another difficulty may occur in tracing the solution, even if a solution satisfying boundary conditions is found. Such a situation is demonstrated in a simple case below.

It is often the case that the initial state is specified, the terminal state is constrained to satisfy a certain equality ($\psi[x(1)] = 0$), and there is no parameter to be optimized. We assume $\dim\psi > 1$. The unknown variables in the stabilized continuation method reduce to the initial costate $\lambda(0)$ and the Lagrange multiplier ν for the terminal constraint in this case. The function F and matrix $[F_z \; F_s]$ are given in this case as follows:

$$F = \begin{bmatrix} Y_x^T - \lambda(1) \\ \psi[x(1)] \end{bmatrix}$$
$$[F_z \quad F_s] = \begin{bmatrix} S_2 & \psi_x^T & : & S_4 \\ \psi_x \Phi_{12}(1) & 0 & : & U_4 \end{bmatrix} \tag{53}$$

If the terminal time is chosen as the continuation parameter s, the terminal state $x(1)$ is identical with the initial state $x(0)$ at $s = 0$. Therefore the terminal constraint cannot be satisfied at $s = 0$ unless $\psi[x(0)] = 0$ holds for the given initial state. One can avoid this difficulty by replacing the function ψ with $\bar{\psi}$ defined as

$$\bar{\psi}[x(1), s] = \psi[x(1; s)] - (1 - s)\psi[x(1; 0)] \tag{54}$$

The terminal constraint $\bar{\psi}[x(1), 0] = 0$ is satisfied at $s = 0$ for any $x(1)$. However, no matter which of functions ψ and $\bar{\psi}$ is employed, $\Phi_{12}(1) = 0$ holds for any $z = [\lambda^T(0) \;\; \nu^T]^T$ when the terminal time is zero. This implies that the matrix $[F_z \; F_s]$ cannot have full rank at $s = 0$, and the zero curve of F cannot be traced by using Eq. (19). This situation occurs since the variation of the initial costate has no effect on the final state.

Similar difficulties as the above case can also occur in other problems with general boundary conditions. In order to avoid difficulties in problems with general boundary conditions, this paper considers two approaches; the multiplier method (augmented Lagrangian method) and the open-loop control. The two approaches can be implemented simply and are applicable to a large class of problems.

B. Multiplier Method in Continuation Method

The multiplier method[22] (augmented Lagrangian method) is an effective technique for converting a constrained minimization problem to an unconstrained one and enables one to use the algorithms for unconstrained minimization. An optimal control problem with the terminal constraint is solved by successively minimizing a series of the performance indices with the terminal penalty replaced by

$$\varphi + \bar{\nu}_{(i)}^T \psi + (c_1/2)\psi^T \psi \tag{55}$$

where $\bar{\nu}_{(i)}$ denotes the multiplier updated by the formula

$$\bar{\nu}_{(i+1)} = \bar{\nu}_{(i)} + c_1 \psi \tag{56}$$

and the scalar c_1 is a positive constant with a sufficiently large magnitude. Note that the penalty term associated with the constraint is added in Eq. (55), which distinguishes the present multiplier method from the simple Lagrange method of multipliers. Although the scalar c_1 may also be updated according to a certain criterion, we assume here it is constant for simplicity. In contrast to the usual multiplier method in which the optimization problem is changed discontinuously as in Eq. (56), the optimization problem is embedded into a family of problems continuously in the present continuation method. Therefore $\bar{\nu}_{(i)}$ is replaced by $\bar{\nu}(s)$, and regarding Eq. (56) as a difference equation, the update formula of the multiplier is modified to a differential equation naturally as

$$\frac{\mathrm{d}\bar{\nu}(s)}{\mathrm{d}s} = c_2 \psi \tag{57}$$

where c_2 is a positive constant. The constant c_2 is not necessarily identical with c_1, since the scaling of $\mathrm{d}\bar{\nu}/\mathrm{d}s$ may be taken arbitrarily with respect to the continuation parameter s. The initial condition at $s = 0$ for Eq. (57) is chosen so that the boundary conditions are satisfied except for the terminal constraint.

The stability of the present continuous multiplier method can be analyzed in a limited situation by using the Lyapunov method. This paper analyzes the stability of the continuous multiplier method for the terminal constraint that does not depend on the continuation parameter explicitly. Further assumptions are that the initial state is given and the free parameter p to be optimized and the continuation parameter s are not included explicitly in the problem formulation. In other words, we consider only the effect of the multiplier method on the process of the continuation method. We have

$$Y = \varphi + \bar{\nu}^T \psi + \tfrac{1}{2} c_1 \psi^T \psi \tag{58}$$

and total differentiation of the terminal condition Eq. (13) with respect to s yields

$$c_2 \psi_x^T \psi = -[Y_{xx}\Phi_{12}(1) - \Phi_{22}(1)]\frac{\delta\lambda(0)}{\mathrm{d}s} \tag{59}$$

where the assumptions $\varphi_s = 0$ and $\psi_s = 0$ are used. We introduce a candidate of the Lyapunov function $\psi^T\psi$, and its derivative with respect to s is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}s}(\psi^T\psi) = 2\psi^T\psi_x \frac{\delta x(1)}{\mathrm{d}s} = -\frac{1}{c_2}\frac{\delta\lambda^T(0)}{\mathrm{d}s} P \frac{\delta\lambda(0)}{\mathrm{d}s} \tag{60}$$

where the matrix P is defined as

$$P = 2\Phi_{12}^T(1) Y_{xx} \Phi_{12}(1) - \Phi_{12}^T(1)\Phi_{22}(1) - \Phi_{22}^T(1)\Phi_{12}(1) \tag{61}$$

The Lyapunov theorem implies that $\psi = 0$ is stable in the sense of Lyapunov if the matrix P is positive semidefinite. Furthermore the asymptotic stability is guaranteed by an extension of the Lyapunov theorem[18] if the set of points at which $d(\psi^T\psi)/\mathrm{d}s$ is zero contains no positive half-trajectory except $\psi = 0$. If $\psi = 0$ is asymptotically stable, the error in the terminal constraint converges to zero as the continuation method proceeds. Therefore the terminal constraint may not be satisfied at the start of the continuation method, which often simplifies formulation of the continuation method. It is observed in Eq. (61) that the matrix Y_{xx} should be so large that P is positive semidefinite. Choosing a large c_1 result in a large Y_{xx}, as is seen from Eq. (58). Consequently, c_1 should be so large that the error in the terminal constraint converges to zero, which is consistent with the fact that the usual multiplier method also requires a sufficiently large weight for the penalty term in order to guarantee the convergence of the solution. An optimal control problem with both the initial and terminal constraints can also be converted to a problem with neither of them by applying the multiplier method to both constraints. Although the stability of the continuous multiplier method is not evident in more general situations than the present analysis, the method works satisfactorily in a variety of problems, as is shown in the numerical example in this paper.

C. Continuation Method with Open-Loop Control

If the above multiplier method is not employed, the terminal time has to be nonzero at $s = 0$ so that the matrix $[F_z \; F_s]$ has full rank, as is shown in the previous analysis. However, it is often difficult to find a solution that satisfies $F = 0$ for a nonzero terminal time. One can start the continuation method without solving $F = 0$ by employing F' in Eq. (20) in place of F or by introducing the open-loop control in Eq. (33), which brings a nonzero F to the origin at a prescribed value of s. Here we consider a case in which the open-loop control in Eq. (33) is introduced. When the matrix A is zero identically in Eq. (32), $\Phi(0, s) = I$, $W(0, s_f) = s_f I$, and the differential equation for F is given by

$$\frac{\mathrm{d}F[z(s), s]}{\mathrm{d}s} = -\frac{1}{s_f}F[z(0), 0] \tag{62}$$

Integration of Eq. (62) yields an equation that is parameterized by a fee parameter $a = z(0)$ as follows:

$$F''(z, s, a) = F(z, s) - (1 - s/s_f)F(a, 0) = 0, \qquad s \in [0, s_f] \tag{63}$$

Therefore the continuation method with the open-loop control is equivalent to tracing the zero curve of F''. The above equation is similar to one employed in a constructive global Newton procedure.[23] It can be shown that Eq. (63) has the same preferable properties as Eq. (20). If the matrix $F_z(a, 0)$ is nonsingular, the parameterized Sard theorem implies that, for almost all a with respect to Lebesgue measure, there is a zero curve of F'' for $s \in [0, s_f)$ along which

$[F''_z \ F''_s]$ has full rank. It can also be shown that if the zero curve of F'' for $s \in [0, s_f)$ is bounded, it has an accumulation point $(\bar{z}, s_f)$, where $F(\bar{z}, s_f) = 0$. Furthermore, if $F_z(\bar{z}, s_f)$ is nonsingular, then the zero curve has finite arc length.[9,13]

A possible difficulty in solving a nonlinear equation associated with TPBVPs is that the numerical integration of the differential equations in the TPBVP fails if the problem is very sensitive to the error in the initial conditions.[3,4,10] The same difficulty may occur in the continuation method with the open-loop control, when $z(s)$ is much different from the solution of $F(z, s) = 0$. In this case, the continuation parameter should be introduced so as to avoid the numerical difficulty. One can avoid overflow in integration of differential equations by increasing the terminal time gradually from a small value.

IV. Numerical Examples

A. Optimal Control Problem

An optimal control problem with initial and terminal constraints is treated here in order to demonstrate the stabilized continuation method. The problem is a variation of the standard problem of maximum velocity transfer to a rectilinear path,[24] which is one of the simplest practical problems in aerospace engineering. A particle of unit mass is acted upon by a thrust force of constant magnitude a. We assume planar motion, and the position and velocity of the particle are expressed by (x_1, x_2) and (x_3, x_4), respectively. The control input u to the system is the thrust direction angle (see Fig. 1). The controlled system is governed by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ a\cos u \\ a\sin u \end{bmatrix} \tag{64}$$

The initial and terminal constraints are given as

$$\chi = \begin{bmatrix} x_1(0) & x_2(0) & x_3^2(0) + x_4^2(0) - V_0^2 \end{bmatrix}^T = 0 \tag{65}$$

$$\psi = [x_2(1) - h \quad x_4(1)]^T = 0 \tag{66}$$

where V_0 denotes the magnitude of the initial velocity and h the final height of the particle, respectively. Note that the direction of the initial velocity is not given, although its magnitude is given. The particle has to reach a given height $x_2 = h$ with the vertical velocity $x_4 = 0$ at the final time $t = 1$. The control objective is to maximize the final horizontal velocity x_3, and the performance index is given as

$$J = -x_3(1) \tag{67}$$

The problem is solved by the stabilized continuation method with the continuation parameter introduced in several ways. The integration intervals are divided into 100 steps for both of t and s in all cases. The parameters are given as $a = 5$, $V_0 = 1$, and $h = 1$ in the numerical solution.

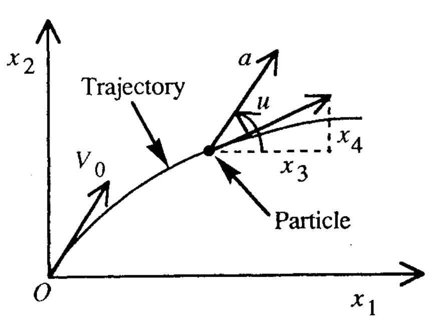

Fig. 1 Planar motion of a particle with acceleration a.

B. Solution by Varying Terminal Time

First, the present optimal control problem is solved by the stabilized continuation method employing the terminal time as the continuation parameter. The terminal constraints are treated through use of the continuous multiplier method. Normalizing the time as $t = s\tau$, the system dynamics is modified so as to include the continuation parameter as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} sx_3 \\ sx_4 \\ as\cos u \\ as\sin u \end{bmatrix}, \qquad 0 \le s,\ \tau \le 1 \tag{68}$$

The TPBVP associated with the parameterized optimal control problem is equivalent to finding the parameterized zero $z(s)$ of the function $F(z, s)$, given as

$$F(z, s) = \begin{bmatrix} 2\mu_3 x_3(0) - 1 \\ 2\mu_3 x_4(0) + \lambda_4(0) \\ \bar{\nu}_1 + c_1[x_2(1) - h] - \lambda_2(1) \\ \bar{\nu}_2 + c_1 x_4(1) - \lambda_4(1) \\ x_3^2(0) + x_4^2(0) - V_0^2 \end{bmatrix}, \qquad z = \begin{bmatrix} x_3(0) \\ x_4(0) \\ \lambda_2(0) \\ \lambda_4(0) \\ \mu_3 \end{bmatrix} \tag{69}$$

where λ_2 and λ_4 denote the costates corresponding to x_2 and x_4 and μ_3 the multiplier corresponding to the initial constraint $\chi_3 = 0$, respectively. The estimates of the multiplier for the terminal constraints are denoted by $\bar{\nu}_1$ and $\bar{\nu}_2$ and are determined by the continuous multiplier method introduced in the previous section. The differential equation for $\bar{\nu}_1$ and $\bar{\nu}_2$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} \bar{\nu}_1 \\ \bar{\nu}_2 \end{bmatrix} = c_2 \begin{bmatrix} x_2(1) - h \\ x_4(1) \end{bmatrix} \tag{70}$$

Since $x_1(0)$ and $x_2(0)$ are set to zero explicitly and some costates can be solved evidently as $\lambda_1(\tau) = 0$ and $\lambda_3(\tau) = -1$, they are not included in the vector of the unknown quantities z.

The initial conditions of z and $\bar{\nu} = [\bar{\nu}_1 \ \bar{\nu}_2]^T$ are determined by giving $x_3(0)$ and $\lambda_2(0)$. Other elements of $z(0)$ and $\bar{\nu}(0)$ are obtained from the conditions $F(z, 0) = 0$, $x_2(1) = x_2(0)$, $\lambda_2(1) = \lambda_2(0)$, and so forth, at $s = 0$. It may be noted that the terminal constraints need not to be satisfied at $s = 0$, since the continuous multiplier method is employed. The initial values of $x_3(0)$ and $\lambda_2(0)$ are given as $x_3(0) = V_0$ and $\lambda_2(0) = 0$ in this example.

The differential equation (23) is integrated with the stabilizing input ν given as the following state feedback:

$$\nu = -\zeta F, \qquad \zeta > 0 \tag{71}$$

In spite of its simple form, previous analysis in Sec. II.B shows that Eq. (71) is an optimal feedback that minimizes the performance index:

$$J_e = \frac{1}{2}\int_0^\infty \left(\zeta^2 F^T F + \nu^T \nu\right) \mathrm{d}s \tag{72}$$

Although the integration interval of s is finite in the present example, the above infinite-horizon criterion Eq. (72) is employed because its optimal control is such a simple state feedback with a constant gain, as in Eq. (71). The parameters of the continuous multiplier method and the stabilizing input are chosen as $c_1 = 20$, $c_2 = 4$, and $\zeta = 100$ in the present case. Trial and error are necessary in choosing the parameters of the multiplier method in order to attain appropriate decay rates of errors in the terminal constraints.

The states x_2 and x_4 are plotted as a function of t and s in Fig. 2 in order to show the history of the solution process of the stabilized continuation method. Note that the terminal time of the optimal control problem is not normalized in Fig. 2. The terminal time is zero at $s = 0$ and is 1 at $s = 1$. The curves $x_2(t, 1)$ and $x_4(t, 1)$ correspond to the desired solutions of the optimal control problem. The curves $x_2(s, s)$ and $x_4(s, s)$ show that the terminal states converge to given values so as to satisfy the terminal constraints. The initial state $x_4(0, s)$ also varies as s increases, being optimized and satisfying

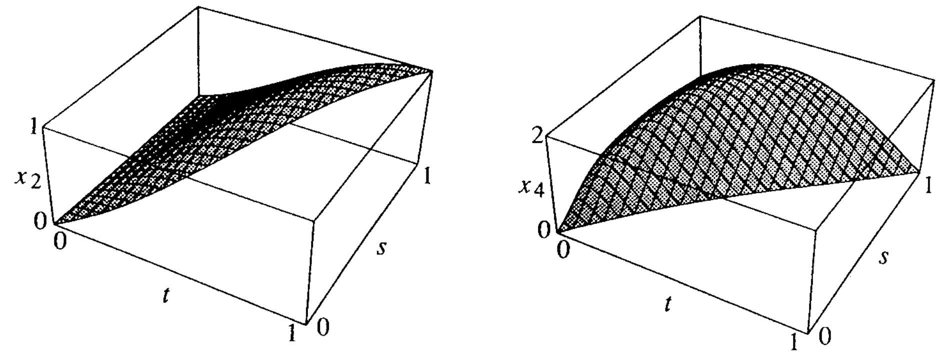

Fig. 2 History of continuation method with respect to terminal time. States $x_2(t, s)$ and $x_4(t, s)$ are plotted.

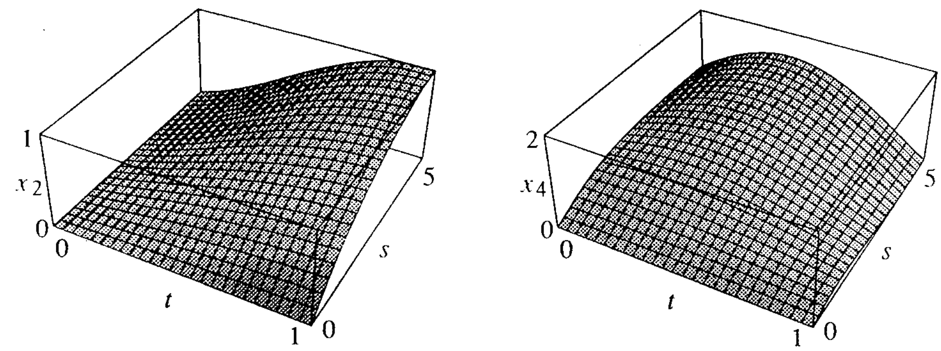

Fig. 3 History of continuation method with respect to control input. States $x_2(t, s)$ and $x_4(t, s)$ are plotted.

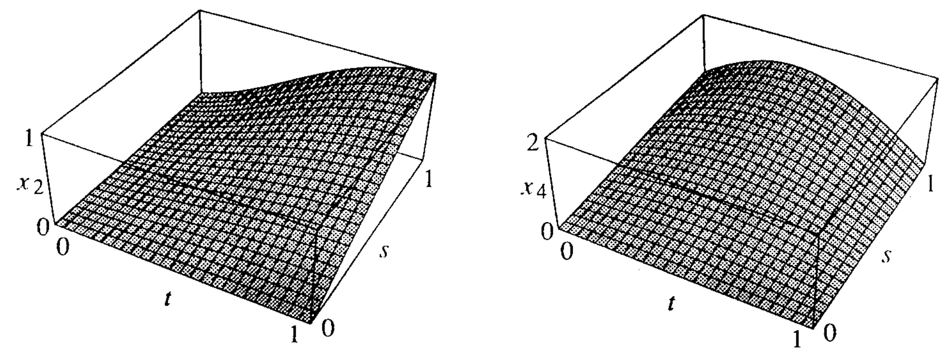

Fig. 4 History of continuation method with tracking control law. States $x_2(t, s)$ and $x_4(t, s)$ are plotted.

the initial constraint. The resultant final velocity $x_3(1, 1)$ is 5.1, and the optimal angle of the initial velocity is obtained as 40°. The orders of the final errors in elements of $F[z(1), 1]$ are less than 10^{-6} except that of F_3, and the order of the error in F_3 is 10^{-2}. The errors are reduced by a factor of $\frac{1}{10}$ to $\frac{1}{100}$ by introducing the stabilizing input in comparison with results without the stabilizing input. The orders of errors in the terminal constraints are less than 10^{-2}, which can be reduced further by executing only the continuous multiplier method after the continuation parameter reaches the final value. It may be concluded that the stabilized continuation method can be implemented with appropriate accuracy for optimal control problems with general boundary conditions.

C. Solution by Continuation in Control Input

Next the optimal control problem is solved by continuation in the control input. The system dynamics in Eq. (64) is modified in this case as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ s\cos u \\ s\sin u \end{bmatrix}, \qquad 0 \le s \le a \tag{73}$$

Note that the effect of the control input u vanishes and the TPBVP is solved trivially at $s = 0$. The nonlinear equation to be solved is the same as the case of varying the terminal time [Eq. (69)], and the continuous multiplier method is employed again in order to treat the terminal constraints. The parameters of the continuous multiplier method and the stabilizing input Eq. (71) are chosen as $c_1 = 4$, $c_2 = 4$, and $\zeta = 1$ in the present case.

The states x_2 and x_4 are plotted as a function of t and s in Fig. 3. Note that the terminal time is one identically from $s = 0$ to $s = a$, which is different from the previous case. The curves $x_2(1, s)$ and $x_4(1, s)$ show that the terminal states converge to given values so as to satisfy the terminal constraints. The initial state $x_4(0, s)$ also varies, being optimized and satisfying the initial constraint. The resultant solution at $s = a$ is same as the previous case. The orders of the final errors in elements of $F[z(1), 1]$ are less than 10^{-6}, and the orders of the final errors in the terminal constraints are 10^{-2}.

D. Solution by Tracking Control Law

As the last example, we examine the continuation method that employs the tracking control law as the stabilizing input. The continuation parameter s is not introduced explicitly in the optimal control problem. The equation to be solved is $F(z) = 0$ with the function $F(z)$ and the vector z given by

$$F(z) = \begin{bmatrix} 2\mu_3 x_3(0) - 1 \\ 2\mu_3 x_4(0) + \nu_1 + \nu_2 \\ x_3^2(0) + x_4^2(0) - V_0^2 \\ x_2(1) - h \\ x_4(1) \end{bmatrix}, \qquad z = \begin{bmatrix} x_3(0) \\ x_4(0) \\ \mu_3 \\ \nu_1 \\ \nu_2 \end{bmatrix} \tag{74}$$

where ν_1 and ν_2 are Lagrange multipliers corresponding to the terminal constraints. Note that the function F and z are different from those in the previous two cases, and the terminal constraints are included in the function F explicitly. The terminal constraints are not necessarily satisfied at $s = 0$ and are controlled to be zero at $s = 1$ by a tracking control law, not by the continuous multiplier method. The differential equation for F in this case is expressed as

$$\frac{dF}{ds} = \nu_{\text{ref}} - \zeta(F - F_{\text{ref}}), \qquad \zeta > 0,\ 0 \le s \le 1 \tag{75}$$

where the references are given by the open-loop control in Sec. III.C as

$$\nu_{\text{ref}}(s) = -F[z(0)] \tag{76}$$

$$F_{\text{ref}}(s) = (1 - s)F[z(0)] \tag{77}$$

The gain of the tracking control is set as $\zeta = 1$ in the numerical solution.

The history of the solution process is shown in Fig. 4. The terminal states $x_2(1, s)$ and $x_4(1, s)$ track each reference well. The orders of the final errors in elements of $F[z(1), 1]$ are less than 10^{-6}. The stabilizing input reduces the final errors by factor of about $\frac{1}{10}$ in comparison with the result without the stabilizing input. One advantage in the stabilized continuation method is improvement in accuracy of solutions without increasing the computational effort significantly.

V. Conclusions

The stabilized continuation method is proposed for solving a large class of optimal control problems with general boundary conditions and nondifferential constraints. The original problem is converted to an initial-value problem for a finite-dimensional ordinary differential equation. Existing integration algorithms are applicable to the initial-value problem. It is shown that a singularity in the differential equation can be avoided by modifying the algorithm. Furthermore, the stabilization techniques are developed to suppress accumulative errors in the integration process efficiently. The stabilization of a correct solution is shown to be equivalent to a control problem of a linear system, and some design procedures of the stabilization term are considered.

The multiplier method (augmented Lagrangian method) is naturally modified in order to give a simple and efficient way to treat terminal constraints and/or initial constraints in the continuation method. An open-loop stabilizing technique is also analyzed and is shown to be applicable to a large class of problems. Numerical examples show that the proposed methods yield solutions of appropriate accuracy for a problem with general boundary conditions. The proposed stabilization techniques improve the accuracy of solutions without increasing the computational effort significantly. The numerical examples also reveal that trial and error are necessary in choosing the parameters of the multiplier method in order to attain appropriate decay rates of errors in terminal constraints. One drawback in the proposed method is that it often requires rather much computation even for such a simple problem as is treated in the example. However, the present approach can be valid for problems that are hard to solve with an iterative method based on successive approximation.

References

[1]Ortega, J. M., and Rheinboldt, W. C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic, New York, 1970, Sec. 7.5.

[2]Richter, S. L., and DeCarlo, R. A., "Continuation Methods: Theory and Applications," *IEEE Transactions on Automatic Control*, Vol. AC-28, No. 6, 1983, pp. 660–665.

[3]Roberts, S. M., and Shipman, J. S., "Continuation in Shooting Methods for Two-Point Boundary Value Problems," *Journal of Mathematical Analysis and Applications*, Vol. 18, 1967, pp. 45–58.

[4]Roberts, S. M., and Shipman, J. S., *Two-Point Boundary-Value Problems: Shooting Methods*, American Elsevier, New York, 1972, Chap. 7.

[5]Orava, P. J., and Lautala, P. A. J., "Interval Length Continuation Method for Solving Two-Point Boundary-Value Problems," *Journal of Optimization Theory and Applications*, Vol. 23, No. 2, 1977, pp. 217–227.

[6]Junkins, J. L., and Turner, J. D., *Optimal Spacecraft Rotational Maneuvers*, Elsevier, Amsterdam, 1986, Sec. 8.5.

[7]Kagiwada, H., Kalaba, R., Rasakhoo, N., and Spingarn, K., "Numerical Experiments Using Sukhanov's Initial-Value Method for Nonlinear Two-Point Boundary-Value Problems, II," *Journal of Optimization Theory and Applications*, Vol. 46, No. 1, 1985, pp. 31–36.

[8]Chow, S.-N., Mallet-Paret, J., and Yorke, J. A., "Finding Zeros of Maps: Homotopy Methods That Are Constructive with Probability One," *Mathematics of Computation*, Vol. 32, No. 143, 1978, pp. 887–899.

[9]Watson, L. T., "An Algorithm That Is Globally Convergent with Probability One for a Class of Nonlinear Two-Point Boundary Value Problems," *SIAM Journal on Numerical Analysis*, Vol. 16, No. 3, 1979, pp. 394–401.

[10]Watson, L. T., "Solving Finite Difference Approximations to Nonlinear Two-Point Boundary Value Problems by a Homotopy Method," *SIAM Journal on Scientific and Statistical Computing*, Vol. 1, No. 4, 1980, pp. 467–480.

[11]Vasudevan, G., Watson, L. T., and Lutze, F. H., "Homotopy Approach for Solving Constrained Optimization Problems," *IEEE Transactions on Automatic Control*, Vol. 36, No. 4, 1991, pp. 494–498.

[12]Rakowska, J., Haftka, R. T., and Watson, L. T., "An Active Set Algorithm for Tracing Parametrized Optima," *Structural Optimization*, Vol. 3, No. 1, 1991, pp. 29–44.

[13]Watson, L. T., "Numerical Linear Algebra Aspects of Globally Convergent Homotopy Methods," *SIAM Review*, Vol. 28, No. 4, 1986, pp. 529–545.

[14]Kabamba, P. T., Longman, R. W., and Jian-Guo, S., "A Homotopy Approach to the Feedback Stabilization of Linear Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 5, 1987, pp. 422–432.

[15]Baumgarte, J., "Stabilization of Constraints and Integrals of Motion in Dynamical Systems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 1, No. 1, 1972, pp. 1–16.

[16]Branin, F. H., Jr., "Widely Convergent Method for Finding Multiple Solutions of Simultaneous Nonlinear Equation," *IBM Journal of Research and Development*, Vol. 16, No. 5, 1972, pp. 504–522.

[17]Kailath, T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980, pp. 610.

[18]Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill, New York, 1970, Sec. 6.7.

[19]Miele, A., Mangiavacchi, A., and Aggarwal, A. K., "Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints," *Journal of Optimization Theory and Applications*, Vol. 14, No. 5, 1974, pp. 529–556.

[20]Gonzalez, S., and Rodriguez, S., "Modified Quasilinearization Algorithm for Optimal Control Problems with Nondifferential Constraints and General Boundary Conditions," *Journal of Optimization Theory and Applications*, Vol. 50, No. 1, 1986, pp. 109–128.

[21]Wu, A. K., and Miele, A., "Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems with Non-Differential Constraints and General Boundary Conditions, Part 1," *Optimal Control Applications & Methods*, Vol. 1, No. 1, 1980, pp. 69–88.

[22]Hestenes, M. R., "Multiplier and Gradient Methods," *Journal of Optimization Theory and Applications*, Vol. 4, No. 5, 1969, pp. 303–320.

[23]Garcia, C. B., and Gould, F. J., "Relations Between Several Path Following Algorithms and Local and Global Newton Methods," *SIAM Review*, Vol. 22, No. 3, 1980, pp. 263–274.

[24]Bryson, A. E., Jr., and Ho, Y.-C., *Applied Optimal Control*, Hemisphere, New York, 1975, Sec. 2.4.